Continuous Penalty Forces
Supplementary Material

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In this supplementary document, we include:

- Proofs of the continuous normal theorems given in the paper.
- A detailed derivation of the polynomial corresponding to the continuous force for linear trajectories.
- A derivation of stability conditions of 1D particle-on-plane contact for Symplectic Euler and continuous penalty forces.

A Time-Varying Contact Normals for VF and EE Pairs

Theorem 1: Continuous Normal Theorem for a Deforming Triangle. *Given the start and end positions of the vertices of a triangle during the interval $[0, 1]$, whose positions are linearly interpolated in the interval with respect to the time variable, t. The unit normal vector, $\vec{n}_T(t)$, of the triangle, at time t, is given by the equation:*

$$\vec{n}_T(t) = \frac{\vec{n}_0\, B_0^2(t) + \vec{n}_1\, B_1^2(t) + \vec{n}_2\, B_2^2(t)}{L(t)}, \tag{1}$$

where

- $B_i^2(t) = \frac{2!}{i!(2-i)!}\, t^i\, (1-t)^{2-i}$.
- $a_0, a_1, b_0, b_1, c_0, c_1$ *are the start and end positions of the three vertices of the deforming triangle, respectively.*
- $\vec{v}_a = a_1 - a_0$, $\vec{v}_b = b_1 - b_0$, *and* $\vec{v}_c = c_1 - c_0$.
- $\vec{n}_0 = (b_0 - a_0) \times (c_0 - a_0)$, $\vec{n}_2 = (b_1 - a_1) \times (c_1 - a_1)$, $\vec{n}_1 = \frac{\vec{n}_0 + \vec{n}_2 - (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_a)}{2}$, *respectively.*
- $L_0 = (\vec{n}_0^T\, \vec{n}_0)$, $L_1 = (\vec{n}_0^T\, \vec{n}_1)$, $L_2 = \frac{2\,(\vec{n}_1^T\, \vec{n}_1) + (\vec{n}_0^T\, \vec{n}_2)}{3}$, $L_3 = (\vec{n}_1^T\, \vec{n}_2)$, $L_4 = (\vec{n}_2^T\, \vec{n}_2)$, *respectively.*
- $B_i^4(t) = \frac{4!}{i!(4-i)!}\, t^i\, (1-t)^{4-i}$.
- $L(t) = \sqrt{(L_0\ L_1\ \ldots\ L_4) \cdot (B_0^4(t)\ B_1^4(t)\ \ldots\ B_4^4(t))^T}$.

Proof. We define the following terms: $\vec{a}_t = \vec{a}_0 + \vec{v}_a\, t$, $\vec{b}_t = \vec{b}_0 + \vec{v}_b\, t$, and $\vec{c}_t = \vec{c}_0 + \vec{v}_c\, t$. The normal vector of triangle $\triangle a_t b_t c_t$ is given as:

$$\begin{aligned} \vec{m}_t &= (\vec{b}_t - \vec{a}_t) \times (\vec{c}_t - \vec{a}_t) \\ &= [(b_0 - a_0) + (\vec{v}_b - \vec{v}_a)\, t] \times [(c_0 - a_0) + (\vec{v}_c - \vec{v}_a)\, t] \\ &= (b_0 - a_0) \times (c_0 - a_0) + (\vec{v}_b - \vec{v}_a) \times (c_0 - a_0)\, t + \\ &\quad (b_0 - a_0) \times (\vec{v}_c - \vec{v}_a)\, t + \\ &\quad (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_a)\, t^2. \end{aligned} \tag{2}$$

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Let $\vec{n}_0$ and $\vec{n}_2$ be the normal vectors of triangle $\triangle a_0 b_0 c_0$ and $\triangle a_1 b_1 c_1$, respectively. Then:

$$\vec{n}_0 = (b_0 - a_0) \times (c_0 - a_0), \tag{3}$$

$$\begin{aligned} \vec{n}_2 &= (b_1 - a_1) \times (c_1 - a_1) \\ &= (b_0 + \vec{v}_b - a_0 - \vec{v}_a) \times (c_0 + \vec{v}_c - a_0 - \vec{v}_a) \\ &= (b_0 - a_0) \times (c_0 - a_0) + (\vec{v}_b - \vec{v}_a) \times (c_0 - a_0) + \\ &\quad (b_0 - a_0) \times (\vec{v}_c - \vec{v}_a) + (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_a). \end{aligned} \tag{4}$$

Based on the above equations, we obtain:

$$\begin{aligned} \vec{n}_2 - \vec{n}_0 &= (\vec{v}_b - \vec{v}_a) \times (c_0 - a_0) + (b_0 - a_0) \times (\vec{v}_c - \vec{v}_a) + \\ &\quad (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_a). \end{aligned} \tag{5}$$

We define:

$$\vec{\omega} = (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_a). \tag{6}$$

Then from equations (5) and (6):

$$\vec{n}_2 - \vec{n}_0 - \vec{\omega} = (\vec{v}_b - \vec{v}_a) \times (c_0 - a_0) + (b_0 - a_0) \times (\vec{v}_c - \vec{v}_a) \tag{7}$$

By plugging the equations (3), (6), and (7) into equation (2), $\vec{m}_t$ can be represented as:

$$\begin{aligned} \vec{m}_t &= \vec{n}_0 + (\vec{n}_2 - \vec{n}_0 - \vec{\omega})\, t + \vec{\omega}\, t^2 \\ &= \vec{n}_0\, (1-t)^2 + \frac{\vec{n}_0 + \vec{n}_2 - \vec{\omega}}{2}\, 2\, t\, (1-t) + \vec{n}_2\, t^2 \\ &= \vec{n}_0\, B_0^2(t) + \frac{\vec{n}_0 + \vec{n}_2 - \vec{\omega}}{2}\, B_1^2(t) + \vec{n}_2\, B_2^2(t) \\ &= \vec{n}_0\, B_0^2(t) + \vec{n}_1\, B_1^2(t) + \vec{n}_2\, B_2^2(t). \end{aligned} \tag{8}$$

And:

$$\begin{aligned} ||\vec{m}_t||^2 &= \vec{m}_t^T\, \vec{m}_t \\ &= (\vec{n}_0^T\, \vec{n}_0)\, B_0^2(t)\, B_0^2(t) \\ &+ 2\,(\vec{n}_0^T\, \vec{n}_1)\, B_0^2(t)\, B_1^2(t) \\ &+ (\vec{n}_1^T\, \vec{n}_1)\, B_1^2(t)\, B_1^2(t) \\ &+ 2\,(\vec{n}_0^T\, \vec{n}_2)\, B_0^2(t)\, B_2^2(t) \\ &+ (\vec{n}_2^T\, \vec{n}_2)\, B_2^2(t)\, B_2^2(t) \\ &+ 2\,(\vec{n}_1^T\, \vec{n}_2)\, B_1^2(t)\, B_2^2(t). \end{aligned} \tag{9}$$

Based on the properties of the Bernstein basis functions, we have:

$$\begin{aligned} B_0^2(t)\, B_0^2(t) &= B_0^4(t) \\ B_0^2(t)\, B_1^2(t) &= \frac{B_1^4(t)}{2} \\ B_1^2(t)\, B_1^2(t) &= \frac{2\, B_2^4(t)}{3} \\ B_0^2(t)\, B_2^2(t) &= \frac{B_2^4(t)}{6} \\ B_1^2(t)\, B_2^2(t) &= \frac{B_3^4(t)}{2} \\ B_2^2(t)\, B_2^2(t) &= B_4^4(t). \end{aligned} \tag{10}$$

By plugging the equation (10) into equation (9), we have:

$$\begin{aligned} ||\vec{m}_t||^2 &= (\vec{n}_0^T\,\vec{n}_0)\,B_0^4(t) + (\vec{n}_0^T\,\vec{n}_1)\,B_1^4(t) \\ &+ \frac{2\,(\vec{n}_1^T\,\vec{n}_1) + (\vec{n}_0^T\,\vec{n}_2)}{3}\,B_2^4(t) \\ &+ (\vec{n}_1^T\,\vec{n}_2)\,B_3^4(t) + (\vec{n}_2^T\,\vec{n}_2)\,B_4^4(t) \\ &= L(t)\,L(t). \end{aligned} \tag{11}$$

Based on equation (8) and (11), the normalized normal vector, $\vec{n}_T(t)$, will be:

$$\begin{aligned} \vec{n}_T(t) &= \frac{\vec{m}_t}{||\vec{m}_t||} \\ &= \frac{\vec{n}_0\,B_0^2(t) + \vec{n}_1\,B_1^2(t) + \vec{n}_2\,B_2^2(t)}{L(t)} \end{aligned} \tag{12}$$

□

Theorem 2: Continuous Normal Theorem for Two Deforming Edges. *Given the start and end positions of the vertices of two edges during the interval* $[0,1]$*, whose positions are linearly interpolated in the interval with respect to the time variable, t, the normal vector,* $\vec{n}_E(t)$*, between the two edges, at time t, is given by the equation:*

$$\vec{n}_E(t) = \frac{\vec{n}'_0\,B_0^2(t) + \vec{n}'_1\,B_1^2(t) + \vec{n}'_2\,B_2^2(t)}{L'(t)}, \tag{13}$$

where

- $B_i^2(t) = \begin{pmatrix} 2 \\ i \end{pmatrix} (1-t)^i\,t^{2-i}$.
- $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$ *are the start and end positions of the four vertices of the two deforming edges, respectively.*
- $\vec{v}_a = a_1 - a_0$, $\vec{v}_b = b_1 - b_0$, $\vec{v}_c = c_1 - c_0$, *and* $\vec{v}_d = d_1 - d_0$.
- $\vec{n}'_0 = (b_0 - a_0) \times (c_0 - d_0)$, $\vec{n}'_2 = (b_1 - a_1) \times (c_1 - d_1)$, $\vec{n}'_1 = \frac{\vec{n}'_0 + \vec{n}'_2 - \vec{\omega}'}{2}$, *and* $\vec{\omega}' = (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_d)$, *respectively.*
- $L'_0 = (\vec{n}_0'^T\,\vec{n}'_0)$, $L'_1 = (\vec{n}_0'^T\,\vec{n}'_1)$, $L'_2 = \frac{2\,(\vec{n}_1'^T\,\vec{n}'_1) + (\vec{n}_0'^T\,\vec{n}'_2)}{3}$, $L'_3 = (\vec{n}_1'^T\,\vec{n}'_2)$, $L'_4 = (\vec{n}_2'^T\,\vec{n}'_2)$, *respectively.*
- $B_i^4(t) = \frac{4!}{i!(4-i)!}\,t^i\,(1-t)^{4-i}$.
- $L'(t) = \sqrt{(L'_0\;L'_1\;\ldots\;L'_4) \cdot (B_0^4(t)\;B_1^4(t)\;\ldots\;B_4^4(t))^T}$.

Proof. We define the following terms: $\vec{a}_t = \vec{a}_0 + \vec{v}_a\,t$, $\vec{b}_t = \vec{b}_0 + \vec{v}_b\,t$, $\vec{c}_t = \vec{c}_0 + \vec{v}_c\,t$, and $\vec{d}_t = \vec{d}_0 + \vec{v}_d\,t$. The normal vector of the two edges defined by a_t, b_t and c_t, d_t, respectively, is given as:

$$\begin{aligned} \vec{m}'_t &= (\vec{b}_t - \vec{a}_t) \times (\vec{c}_t - \vec{d}_t) \\ &= [(b_0 - a_0) + (\vec{v}_b - \vec{v}_a)\,t] \times [(c_0 - d_0) + (\vec{v}_c - \vec{v}_d)\,t] \\ &= (b_0 - a_0) \times (c_0 - d_0) + (\vec{v}_b - \vec{v}_a) \times (c_0 - d_0)\,t + \\ &\quad (b_0 - a_0) \times (\vec{v}_c - \vec{v}_d)\,t + \\ &\quad (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_d)\,t^2. \end{aligned} \tag{14}$$

Let $\vec{n}'_0$ and $\vec{n}'_2$ be the normal vectors of the two edges defined by a_0, b_0, c_0, d_0 and a_1, b_1, c_1, d_1, respectively. Then:

$$\vec{n}'_0 = (b_0 - a_0) \times (c_0 - d_0), \tag{15}$$

$$\begin{aligned} \vec{n}'_2 &= (b_1 - a_1) \times (c_1 - d_1) \\ &= (b_0 + \vec{v}_b - a_0 - \vec{v}_a) \times (c_0 + \vec{v}_c - d_0 - \vec{v}_d) \\ &= (b_0 - a_0) \times (c_0 - d_0) + (\vec{v}_b - \vec{v}_a) \times (c_0 - d_0) + \\ &\quad (b_0 - a_0) \times (\vec{v}_c - \vec{v}_d) + (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_d). \end{aligned} \tag{16}$$

Based on the above equations, we obtain:

$$\begin{aligned} \vec{n}'_2 - \vec{n}'_0 &= (\vec{v}_b - \vec{v}_a) \times (c_0 - d_0) + (b_0 - a_0) \times (\vec{v}_c - \vec{v}_d) + \\ &\quad (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_d). \end{aligned} \tag{17}$$

We define:

$$\vec{\omega}' = (\vec{v}_b - \vec{v}_a) \times (\vec{v}_c - \vec{v}_d). \tag{18}$$

Then from equations (17) and (18):

$$\vec{n}'_2 - \vec{n}'_0 - \vec{\omega}' = (\vec{v}_b - \vec{v}_a) \times (c_0 - d_0) + (b_0 - a_0) \times (\vec{v}_c - \vec{v}_d) \tag{19}$$

By plugging the equations (15), (18), and (19) into equation (14), $\vec{m}'_t$ can be represented as:

$$\begin{aligned} \vec{m}'_t &= \vec{n}'_0 + (\vec{n}'_2 - \vec{n}'_0 - \vec{\omega}')\,t + \vec{\omega}'\,t^2 \\ &= \vec{n}'_0\,(1-t)^2 + \frac{\vec{n}'_0 + \vec{n}'_2 - \vec{\omega}'}{2}\,2\,t\,(1-t) + \vec{n}'_2\,t^2 \\ &= \vec{n}'_0\,B_0^2(t) + \frac{\vec{n}'_0 + \vec{n}'_2 - \vec{\omega}'}{2}\,B_1^2(t) + \vec{n}'_2\,B_2^2(t) \\ &= \vec{n}'_0\,B_0^2(t) + \vec{n}'_1\,B_1^2(t) + \vec{n}'_2\,B_2^2(t) \end{aligned} \tag{20}$$

Analogous to the deduction in Theorem 1, the normalized normal vector, $\vec{n}_E(t)$, will be:

$$\begin{aligned} \vec{n}_E(t) &= \frac{\vec{m}'_t}{||\vec{m}'_t||} \\ &= \frac{\vec{n}'_0\,B_0^2(t) + \vec{n}'_1\,B_1^2(t) + \vec{n}'_2\,B_2^2(t)}{\sqrt{(L'_0\;L'_1\;\ldots\;L'_4) \cdot (B_0^4(t)\;B_1^4(t)\;\ldots\;B_4^4(t))^T}} \\ &= \frac{\vec{n}'_0\,B_0^2(t) + \vec{n}'_1\,B_1^2(t) + \vec{n}'_2\,B_2^2(t)}{L'(t)} \end{aligned} \tag{21}$$

□

B Computation of Continuous Penalty Force

In this section, we give the exact formula for the degree six polynomial used for contact force computation. We use Corollary I to derive this formula for VF contact force.

Coefficients of the Degree-Six Polynomial. For the evaluation of the following equation:

$$\vec{I}_p = k \sum_{i=0}^{i<N} \int_{t_a^i}^{t_b^i} (\vec{n}_T)^T (\vec{p} - w_a\,\vec{a} - w_b\,\vec{b} - w_c\,\vec{c})\,\vec{n}_T\,dt,$$

where t is the unknown. $\vec{n}_T$ is approximated by a quadratic polynomial (by replacing $L(t)$ with L_k). $\vec{p}, \vec{a}, \vec{b}, \vec{c}$ are linear polynomials; w_a, w_b, and w_c are scalars.

From Theorem 1, let:

$$\begin{aligned} \vec{n}_T &= \hat{a}t^2 + \hat{b}t + \hat{c}, \\ \vec{p} - w_a\,\vec{a} - w_b\,\vec{b} - w_c\,\vec{c} &= \hat{d}t + \hat{e}, \end{aligned}$$

where:

$$\begin{aligned} \hat{a} &= \frac{\vec{n}_0 - 2\vec{n}_1 + \vec{n}2}{L_k}, \\ \hat{b} &= \frac{2(\vec{n}_1 - \vec{n}_0)}{L_k}, \\ \hat{c} &= \frac{\vec{n}_0}{L_k}, \\ \hat{d} &= p_0 - w_a a_0 - w_b b_0 - w_c c_0, \\ \hat{e} &= (p_1 - p0) - w_a(a_1 - a_0) - w_b(b_1 - b_0) - w_c(c_1 - c_0). \end{aligned}$$

The inner term of the integral corresponds to a degree-five polynomial:

$$(\vec{n}_T)^T (\vec{p} - w_a\, \vec{a} - w_b\, \vec{b} - w_c\, \vec{c})\, \vec{n}_T = \hat{a}' t^5 + \hat{b}' t^4 + \hat{c}' t^3 + \hat{d}' t^2 + \hat{e}' t + \hat{f}',$$

where:

$$\begin{aligned}
\hat{a}' &= (\hat{a}^T\, \hat{d})\, \hat{a}, \\
\hat{b}' &= (\hat{a}^T\, \hat{d})\, \hat{b} + (\hat{a}^T\, \hat{e} + \hat{b}^T\, \hat{d})\, \hat{a}, \\
\hat{c}' &= (\hat{a}^T\, \hat{d})\, \hat{c} + (\hat{a}^T\, \hat{e} + \hat{b}^T\, \hat{d})\, \hat{b} + (\hat{b}^T\, \hat{e} + \hat{c}^T\, \hat{d})\, \hat{a}, \\
\hat{d}' &= (\hat{a}^T\, \hat{e} + \hat{b}^T\, \hat{d})\, \hat{c} + (\hat{b}^T\, \hat{e} + \hat{c}^T\, \hat{d})\, \hat{b}, \\
\hat{e}' &= (\hat{b}^T\, \hat{e} + \hat{c}^T\, \hat{d})\, \hat{c} + (\hat{c}^T\, \hat{e})\, \hat{b}, \\
\hat{f}' &= (\hat{c}^T\, \hat{e})\, \hat{c}.
\end{aligned}$$

Then we get the coefficients of the degree-six polynomial:

$$\int_{t_a^i}^{t_b^i} (\vec{n}_T)^T (\vec{p} - w_a\, \vec{a} - w_b\, \vec{b} - w_c\, \vec{c})\, \vec{n}_T\, dt =$$
$$\left(\frac{\hat{a}' t^6}{6} + \frac{\hat{b}' t^5}{5} + \frac{\hat{c}' t^4}{4} + \frac{\hat{d}' t^3}{3} + \frac{\hat{e}' t^2}{2} + \hat{f}' t\right)\Big|_{t_a^i}^{t_b^i}.$$

Analog to VF contact force, we can similarly derive the coefficients of the degree six polynomial for EE contact force.

C Stability Analysis of 1D Particle-on-Plane Contact

The (continuous) motion of a particle with mass m under the action of gravity and a penalty force with stiffness k centered at $x = 0$ is described, through Newton's 2^{nd} Law, as:

$$m\, \dot{v} = -m\, g - k\, x. \tag{22}$$

To analyze the stability of an integration method, we discretize the equation above with a time step Δt, discard the gravity force term, and write an iterative update rule of the form

$$\begin{pmatrix} x(t+\Delta t) \\ v(t+\Delta t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{23}$$

The integration method is stable for time steps for which all eigenvalues $\|\lambda(A)\| < 1$.

With Symplectic Euler (SE), the velocity and position updates can be written as

$$v(t+\Delta t) = v(t) - \Delta t\, \frac{k}{m}\, x(t), \tag{24}$$
$$x(t+\Delta t) = x(t) + \Delta t\, v(t+\Delta t). \tag{25}$$

In the form of equation 23, the update rule is

$$\begin{pmatrix} x(t+\Delta t) \\ v(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 1 - \Delta t^2 \frac{k}{m} & \Delta t \\ -\Delta t \frac{k}{m} & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{26}$$

And the eigenvalue analysis yields $\Delta t < 2\sqrt{\frac{m}{k}}$ for stability.

With Continuous Penalty Forces (CPF), following the formulation in Section 3.3 in the paper, we first predict the velocity at the end of the time step, which in this case is simply $v^*(t+\Delta t) = v(t)$. Then, we integrate the penalty force, and we obtain the following average force:

$$F^* = \frac{1}{\Delta t} \int_0^{\Delta t} -k\, (x(t) + \tau\, v(t))\, d\tau = -k\, x(t) - \Delta t \frac{k}{2}\, v(t). \tag{27}$$

The velocity and position updates with CPF can be written as

$$v(t+\Delta t) = v(t) - \Delta t\, \frac{k}{m}\, x(t) - \Delta t^2 \frac{k}{2\, m}\, v(t), \tag{28}$$
$$x(t+\Delta t) = x(t) + \Delta t\, v(t+\Delta t). \tag{29}$$

In the form of equation 23, the update rule is

$$\begin{pmatrix} x(t+\Delta t) \\ v(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 1 - \Delta t^2 \frac{k}{m} & \Delta t - \Delta t^3 \frac{k}{2m} \\ -\Delta t \frac{k}{m} & 1 - \Delta t^2 \frac{k}{2m} \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{30}$$

And the eigenvalue analysis yields $\Delta t < \sqrt{2\frac{m}{k}}$ for stability.